

THE SCHUR-HORN THEOREM FOR OPERATORS WITH THREE POINT SPECTRUM

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ABSTRACT. We characterize the set of diagonals of the unitary orbit of a self-adjoint operator with three points in the spectrum. Our result gives a Schur-Horn theorem for operators with three point spectrum analogous to Kadison's result for orthogonal projections [14, 15].

1. INTRODUCTION

The goal of this paper is to establish an analogue of the Schur-Horn theorem on an infinite dimensional separable Hilbert space for operators with three points in the spectrum. That is, we will give necessary and sufficient conditions for a countable sequence $\{d_i\}$ to be the diagonal of a self-adjoint operator with eigenvalues $\{0, A, B\}$ with specified (possibly infinite) multiplicities.

This paper falls into a broader category of research that aims at finding an analogue of the Schur-Horn theorem for operators on a separable infinite dimensional Hilbert space. Recently there has been a great deal of progress by a number of authors. The work of Gohberg and Markus [12] and Arveson and Kadison [5] extended the Schur-Horn theorem to positive trace class operators. More recently Kaftal and Weiss [16] have extended this to all positive compact operators. Antezana, Massey, Ruiz, and Stojanoff [1] established a connection between this problem and frame theory, as well as establishing some necessary and sufficient conditions. See [8, 9, 10, 17] for more on this problem from a frame theory perspective. Other work in this area includes the study of II_1 factors by Argerami and Massey [2, 3] and normal operators by Arveson [4]. Neumann [18] proved what may be considered an approximate Schur-Horn theorem since it is given in terms of the ℓ^∞ -closure of the set of diagonal sequences. Bownik and the author [7] established a variant of the Schur-Horn theorem for the set of locally invertible positive operators.

Of particular interest for our purposes is Kadison's theorem [14, 15]. For any orthogonal projection P , this theorem gives an explicit characterization of the set of diagonal sequences of the unitary orbit of P . This can be considered as an infinite dimensional extension of the Schur-Horn theorem for operators with two points in the spectrum. It is a natural next step to consider operators with three points in the spectrum. In this paper we extend the Schur-Horn theorem to such operators.

We would like to emphasize two significant qualitative differences between Kadison's theorem and our extension to operators with three point spectrum. The necessary and sufficient

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condition for a sequence to be the diagonal of a projection is a single trace condition, that is an equation involving sums of diagonal terms. The requirements for a sequence to be the diagonal of an operator with three point spectrum involve both a trace condition and a majorization inequality.

Also distinct from the case of operators with two point spectrum, it is possible for two non-unitarily equivalent operators with three point spectrum to have the same diagonal. For projections the dimension of the kernel and range (i.e. the multiplicities of 0 and 1) can be recovered from the diagonal. Indeed, if $\{d_i\}$ is the diagonal of a projection P , then

$$\dim \operatorname{ran} P = \sum d_i \quad \text{and} \quad \dim \ker P = \sum (1 - d_i).$$

However, for operators with three point spectrum the multiplicities cannot in general be determined from the diagonal, see Remark 5.2.

This leads to two distinct extensions of the Schur-Horn theorem for operators with three point spectrum. In the case where the multiplicities of eigenvalues are not given we have the following general theorem characterizing diagonals of operators with three point spectrum.

3pt **Theorem 1.1.** *Let $0 < A < B < \infty$ and let $\{d_i\}_{i \in I}$ be a countable sequence in $[0, B]$ with $\sum d_i = \sum (B - d_i) = \infty$. Define*

CandD (1.1)
$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a positive operator E with diagonal $\{d_i\}_{i \in I}$ and $\sigma(E) = \{0, A, B\}$ if and only if one of the following holds:

- (i) $C = \infty$
- (ii) $D = \infty$
- (iii) $C, D < \infty$ and there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

3pttrace (1.2)
$$C - D = NA + kB$$

3ptmaj (1.3)
$$C \geq (N + k)A.$$

The assumption that $\sum d_i = \sum (B - d_i) = \infty$ is not a true limitation. Indeed, the summable case $\sum d_i < \infty$ requires more restrictive conditions which can be deduced from parts (a) and (b) of Theorem 1.2. For the proof of Theorem 1.1 we refer the reader to Theorem 5.2.

Theorem 1.2 is our second extension of the Schur-Horn theorem which gives a complete list of characterization conditions of diagonals of operators with prescribed multiplicities. Before we state the full theorem, we need one convenient definition.

Definition 1.1. Let E be a bounded operator on a Hilbert space. For $\lambda \in \mathbb{C}$ define

$$m_E(\lambda) = \dim \ker(E - \lambda).$$

fullthm **Theorem 1.2.** *Suppose $0 < A < B < \infty$, let $\{d_i\}_{i \in I}$ be a countable (possibly finite) sequence in $[0, B]$, and suppose $N, K, Z \in \mathbb{N}$. Define the sets*

$$I_1 = \{i \in I : d_i < A\}, \quad I_2 = \{i \in I : d_i \geq A\}, \quad J_2 = \{i \in I_2 : d_i < (A + B)/2\}, \quad J_3 = I_2 \setminus J_2.$$

Let C and D be as in (1.1) and define the constants (each possibly infinite)

$$C_1 = \sum_{i \in I_1} (A - d_i), \quad C_2 = \sum_{i \in J_2} (d_i - A), \quad C_3 = \sum_{i \in J_3} (B - d_i).$$

The following table gives necessary and sufficient conditions for $\{d_i\}$ to be the diagonal of a positive operator E with $\sigma(E) = \{0, A, B\}$ and the specified multiplicities.

	$m_E(0)$	$m_E(A)$	$m_E(B)$	Condition
(a)	Z	N	K	$ I = Z + N + K$ $\sum_{i \in I} d_i = NA + KB, \quad C \geq (N + K - I_2)A$
(b)	∞	N	K	$ I_1 = \infty,$ $\sum_{i \in I} d_i = NA + KB, \quad C \geq (N + K - I_2)A$
(c)	∞	∞	∞	$C + D = \infty$
(d)	∞	N	∞	$C + D = \infty$ <i>or</i> $C, D < \infty, \quad I_1 = I_2 = \infty,$ $\exists k \in \mathbb{Z} \text{ such that } C - D = NA + kB, \quad C \geq A(N + k)$
(e)	Z	∞	∞	$C_1 \leq AZ, \quad C_2 + C_3 = \infty$ <i>or</i> $ I_1 \cup J_2 = J_3 = \infty, \quad C_1 \leq AZ, \quad C_2, C_3 < \infty$ $\exists k \in \mathbb{Z} \text{ such that } C_1 - C_2 + C_3 = (Z - k)A + kB$
(f)	Z	∞	K	$ I = \infty, \quad C_1 \leq AZ$ $\sum_{i \in I} (d_i - A) = K(B - A) - ZA$

Note that in the preceding theorem we left out the case where only B has infinite multiplicity and the case where only B has finite multiplicity. However, these two remaining cases follow easily using symmetry arguments by applying parts (b) and (e) to the operator $BI - E$ and the sequence $\{B - d_i\}$. Also, observe that case (a) corresponds to the finite dimensional case, and hence it is the classical Schur-Horn theorem (for operators with three eigenvalues), albeit written in a new form. Finally, in this paper we only consider the case of separable Hilbert spaces, and thus the indexing set I is always taken to be a countable (possibly finite) set. We will use the notation C and D given in (1.1) as well as the notation introduced in Theorem 1.2 throughout the rest of the paper.

The proof of Theorem 1.2 breaks into 4 distinct parts. The summable cases (a) and (b) do not require many new techniques since they reduce to the study of trace class operators. In Section 3 parts (a) and (b) are relatively easily deduced from the work of Arveson-Kadison [5]. The remaining 3 parts rely heavily on a technique, which was introduced in [7], of “moving” diagonal entries to more favorable configurations (see Lemma 4.3), where it is possible to construct required operators. In Section 4 we deal with the case (c) involving three (or more) eigenvalues of infinite multiplicity. The necessity of the condition in (c) follows from Theorem 4.1, the sufficiency follows from Theorem 4.2. Much more involved

combinatorial arguments are needed in Section 5 to deal with case (d) involving two outer eigenvalues with infinite multiplicities. Part (d) is proved in Theorem 5.1. In Section 6 we analyze the cases (e) and (f) where at least one of outer eigenvalues has finite multiplicity. The proofs of the necessity and the sufficiency in these last two cases require even more subtle combinatorial arguments which is partially evidenced by the complicated nature of the characterization conditions. In Theorem 6.2 we show that the conditions in part (e) are necessary, while in Theorem 6.3 we show that they are sufficient. Finally, part (f) of Theorem 1.2 is proved in Corollary 6.4.

We finish the paper with an application of Theorem 1.2 in Section 7. Given a sequence $\{d_i\}$ in $[0, 1]$ we are interested in determining the set of numbers $A \in (0, 1)$ for which there exists a positive operator with spectrum $\{0, A, 1\}$ and diagonal $\{d_i\}$. We show that this set is either finite or the full open interval $(0, 1)$. Finally, we look at some specific sequences $\{d_i\}$ and explicitly calculate the set of possible A .

2. PRELIMINARIES

Our arguments rely on the classical Schur-Horn theorem [13, 19], which we state here.

[sh] **Theorem 2.1** (Schur, Horn). *Let $N \in \mathbb{N}$ and let $\{\lambda_i\}_{i=1}^N$ and $\{d_i\}_{i=1}^N$ be nonincreasing sequences of real numbers. There is an $N \times N$ self-adjoint matrix with eigenvalues $\{\lambda_i\}$ and diagonal $\{d_i\}$ if and only if*

[fmaj] (2.1)
$$\begin{aligned} \sum_{i=1}^n d_i &\leq \sum_{i=1}^n \lambda_i \quad \text{for all } n \leq N \\ \sum_{i=1}^N d_i &= \sum_{i=1}^N \lambda_i. \end{aligned}$$

In fact, we need a version of the Schur-Horn theorem for finite rank operators. This can be deduced from a theorem of Arveson and Kadison [5, Theorem 4.1] or from a theorem of Kaftal and Weiss [16, Theorem 6.1].

[frsh] **Theorem 2.2** (Arveson, Kadison). *Let $\{\lambda_i\}_{i=1}^N$ be positive and nonincreasing. Let $\{d_i\}_{i=1}^\infty$ be nonnegative and nonincreasing. There is a rank N positive operator with positive eigenvalues $\{\lambda_i\}$ and diagonal $\{d_i\}$ if and only if*

[frmaj] (2.2)
$$\begin{aligned} \sum_{i=1}^n d_i &\leq \sum_{i=1}^n \lambda_i \quad \text{for all } n \leq N \\ \sum_{i=1}^\infty d_i &= \sum_{i=1}^N \lambda_i. \end{aligned}$$

We will also make extensive use of Kadison's theorem [14, 15].

[Kadison] **Theorem 2.3** (Kadison). *Let $\{d_i\}_{i \in I}$ be a sequence in $[0, 1]$ and $\alpha \in (0, 1)$. Define*

$$a = \sum_{d_i < \alpha} d_i \quad b = \sum_{d_i \geq \alpha} (1 - d_i).$$

There is a projection with diagonal $\{d_i\}_{i \in I}$ if and only if

[kadcond] (2.3)
$$a - b \in \mathbb{Z} \cup \{\pm\infty\},$$

with the convention that $\infty - \infty = 0$.

Remark 2.1. Observe that in Theorem 2.3, if there exists a partition of $I = I_1 \cup I_2$ such that

$$\sum_{i \in I_1} d_i, \sum_{i \in I_2} (1 - d_i) < \infty, \quad \text{and} \quad \sum_{i \in I_1} d_i - \sum_{i \in I_2} (1 - d_i) \in \mathbb{Z},$$

then we have $a - b \in \mathbb{Z}$ for all $\alpha \in (0, 1)$. Thus, the existence of such a partition is also a sufficient condition for a sequence to be the diagonal of a projection. We will find use for these more general partitions in the sequel.

3. FINITE RANK OPERATORS

The following is an application of Theorems 2.1 and 2.2, which establishes parts (a) and (b) of Theorem 1.2. Theorem 3.1 is the analogue of Theorem 3.4 from [7] which characterizes the diagonals of finite rank operators such that $\{A, B\} \subset \sigma(E) \subset \{0\} \cup [A, B]$.

fr3pt **Theorem 3.1.** *Let $0 < A < B < \infty$, let $\{d_i\}_{i \in I}$ be a summable sequence in $[0, B]$, and let $N, K \in \mathbb{N}$ with $N + K < |I|$. There is a positive rank $N + K$ operator E with diagonal $\{d_i\}$, $\sigma(E) = \{0, A, B\}$, $m_E(A) = N$, and $m_E(B) = K$ if and only if*

fr3pt1 (3.1)
$$\sum_{i \in I} d_i = NA + KB$$

fr3pt2 (3.2)
$$\sum_{d_i < A} d_i \geq (N + K - n_0)A,$$

where $n_0 = |\{i : d_i \geq A\}|$.

Proof. We will first prove the theorem under the assumption that $\{d_i\}$ can be arranged in nonincreasing order. Setting $M = |I| \in \mathbb{N} \cup \{\infty\}$, we may assume our sequence is given by $\{d_i\}_{i=1}^M$ in nonincreasing order.

To prove that (3.1) and (3.2) are necessary, assume E is a positive operator with diagonal $\{d_i\}_{i \in I}$, $\sigma(E) = \{0, A, B\}$, $m_E(A) = N$ and $m_E(B) = K$. The operator E has finite rank, hence it is of trace class with trace equal to $NA + KB$; this is (3.1). The eigenvalues sequence of E written in nonincreasing order is given by

fr3pt4 (3.3)
$$\lambda_i = \begin{cases} B & i = 1, 2, \dots, K \\ A & i = K + 1, \dots, K + N \\ 0 & i > K + N. \end{cases}$$

Using Theorem 2.1 (or Theorem 2.2 if $|I| = \infty$) we see that

fr3pt3 (3.4)
$$\sum_{d_i \geq A} d_i = \sum_{i=1}^{n_0} d_i \leq \sum_{i=1}^{n_0} \lambda_i \leq KB + (n_0 - K)A.$$

To see the last inequality in (3.4), consider separately the cases where $n_0 \geq K$ and $n_0 < K$. Using (3.4) we have

$$\sum_{d_i < A} d_i = NA + KB - \sum_{d_i \geq A} d_i \geq NA + KB - (KB + (n_0 - K)A) = A(N + K - n_0),$$

which is (3.2).

Next, assume that (3.1) and (3.2) hold. Define the sequence $\{\lambda_i\}_{i=1}^M$ as in (3.3). By Theorem 2.1 or Theorem 2.2 it is enough to show that

$$\boxed{\text{eq1}} \quad (3.5) \quad \sum_{i=1}^m d_i \leq \sum_{i=1}^m \lambda_i$$

for all $m \leq M$, since the second condition in either (2.1) or (2.2) follows from the assumption (3.1). Note that (3.5) holds for $m \leq K$, since $d_i \leq B$ for all $i \in I$. For $m > K + N$ we have

$$\sum_{i=1}^m d_i \leq \sum_{i=1}^M d_i = \sum_{i=1}^M \lambda_i = \sum_{i=1}^m \lambda_i,$$

so (3.5) holds for $m > K + N$.

First, we wish to show that (3.5) holds for $m = n_0$. From the above we may assume $K < n_0 \leq K + N$. Using (3.2) we have

$$\sum_{i=1}^{n_0} d_i = NA + KB - \sum_{d_i < A} d_i \leq NA + KB - A(N + K - n_0) = KB + (n_0 - K)A = \sum_{i=1}^{n_0} \lambda_i.$$

Now, if $K < m < n_0$ then we have

$$\sum_{i=1}^m d_i = \sum_{i=1}^{n_0} d_i - \sum_{i=m+1}^{n_0} d_i \leq \sum_{i=1}^{n_0} \lambda_i - (n_0 - m)A = \sum_{i=1}^m \lambda_i.$$

Finally, if $n_0 < m \leq K + N$ then

$$\sum_{i=1}^m d_i = \sum_{i=1}^{n_0} d_i + \sum_{i=n_0+1}^m d_i \leq \sum_{i=1}^{n_0} \lambda_i + (m - n_0)A = \sum_{i=1}^m \lambda_i.$$

To complete the proof we assume $\{d_i\}$ cannot be arranged in nonincreasing order. This is the case exactly when $\{d_i\}$ has infinitely many nonzero terms and some terms equal to zero.

Assume we have an operator E with diagonal $\{d_i\}$, $\sigma(E) = \{0, A, B\}$, $m_E(A) = N$ and $m_E(B) = K$. Let $\{e_i\}_{i \in I}$ be an orthonormal basis such that $d_i = \langle E e_i, e_i \rangle$ for each $i \in I$. Set $I_0 = \{i \in I : d_i = 0\}$. Since E is positive, $e_i \in \ker E$ for each $i \in I_0$, and thus $\overline{\text{span}}\{e_i\}_{i \in I \setminus I_0}$ is invariant under E . Let E' be E acting on the space $\overline{\text{span}}\{e_i\}_{i \in I \setminus I_0}$. The operators E and E' have the same multiplicities at A and B , and E' has diagonal $\{d_i\}_{i \in I \setminus I_0}$. The diagonal of E' is a strictly positive summable sequence, and thus it can be arranged in nonincreasing order. By the above argument, we see that (3.1) and (3.2) hold for $\{d_i\}_{i \in I \setminus I_0}$. Clearly this implies that they hold for the full sequence $\{d_i\}_{i \in I}$.

Finally, assume that (3.1) and (3.2) hold. The sequence $\{d_i\}_{i \in I \setminus I_0}$ also satisfies (3.1) and (3.2). Moreover, $\{d_i\}_{i \in I \setminus I_0}$ can be arranged in nonincreasing order. By the above argument, there is a positive operator E' with diagonal $\{d_i\}_{i \in I \setminus I_0}$, $\sigma(E') \subset \{0, A, B\}$, $m_{E'}(A) = N$, and $m_{E'}(B) = K$. Let $\mathbf{0}$ be the zero operator on a separable Hilbert space with dimension $|I_0|$. The operator $E = E' \oplus \mathbf{0}$ has the desired spectral properties and diagonal. \square

4. THREE OR MORE EIGENVALUES OF INFINITE MULTIPLICITY

In this section we will classify the diagonals of operators with exactly three eigenvalues, each with infinite multiplicity. This will yield part (c) of Theorem 1.2. We will also show that a sequence with $C + D = \infty$ is the diagonal of a very general class of operators.

Theorem 4.1 is the analogue of [7, Theorem 5.1], and it is used in showing the necessity of part (c) of Theorem 1.2. In particular, Theorem 4.1 shows that $C, D < \infty$ implies that only 0 and B can have infinite multiplicity. Thus, $C + D = \infty$ is a necessary condition for a sequence to be the diagonal of a self-adjoint operator with at least three infinite multiplicities.

nec4 **Theorem 4.1.** *Let $0 < A < B < \infty$ and let E be a positive operator on a Hilbert space \mathcal{H} with $\sigma(E) = \{0, A, B\}$. Let $\{e_i\}_{i \in I}$ be an orthonormal basis for \mathcal{H} and set $d_i = \langle E e_i, e_i \rangle$. If $C, D < \infty$ then $N := m_E(A) < \infty$ and there is some $k \in \mathbb{Z}$ such that*

nec4.1 (4.1)
$$C - D = NA + kB,$$

nec4.5 (4.2)
$$C \geq (N + k)A.$$

Proof. Define the sets $I_1 = \{i : d_i < A\}$ and $I_2 = \{i : d_i \geq A\}$. Let P be the orthogonal projection onto $\ker(E - A)$, and let Q be the projection onto $\ker(E - B)$. This yields the decomposition $E = AP + BQ$. Define $p_i = \langle P e_i, e_i \rangle$ and $q_i = \langle Q e_i, e_i \rangle$, so that $d_i = Ap_i + Bq_i$. By [7, Theorem 5.1], the operator $B(Q + P) - E = (B - A)P$ is of trace class and thus finite rank. From this we conclude

nec4.2 (4.3)
$$N = m_E(A) = \sum_{i \in I} p_i < \infty.$$

Define

$$a := \sum_{i \in I_1} q_i = \frac{1}{B} \sum_{i \in I_1} (d_i - Ap_i) \leq \frac{1}{B} \sum_{i \in I_1} d_i = \frac{C}{B} < \infty,$$

and

$$b := \sum_{i \in I_2} (1 - q_i) = \frac{1}{B} \sum_{i \in I_2} (B - d_i + Ap_i) \leq \frac{D}{B} + \frac{A}{B} \sum_{i \in I_2} p_i.$$

Using (4.3) we see that $b < \infty$. By Theorem 2.3 there exists $k \in \mathbb{Z}$ such that $a - b = k$.

Now, we calculate

$$\begin{aligned} C - D &= \sum_{i \in I_1} (Ap_i + Bq_i) - \sum_{i \in I_2} (B - Ap_i - Bq_i) \\ &= \sum_{i \in I} Ap_i + B \left(\sum_{i \in I_1} q_i - \sum_{i \in I_2} (1 - q_i) \right) = NA + kB, \end{aligned}$$

which shows (4.1).

Finally, we calculate

$$\begin{aligned} k(B - A) + D &= (a - b)(B - A) + \sum_{i \in I_2} (B - Bq_i - Ap_i) = a(B - A) + bA - \sum_{i \in I_2} Ap_i \\ &\geq bA - bB + bB - \sum_{i \in I_2} Ap_i = bA - \sum_{i \in I_2} Ap_i = A \sum_{i \in I_2} (1 - p_i - q_i). \end{aligned}$$

Together with the fact that $p_i + q_i \leq 1$, this shows $k(B - A) + D \geq 0$, or $kB + D \geq kA$. Combining this with (4.1) gives (4.2). \square

Next, we will show that the condition $C + D = \infty$ is sufficient for $\{d_i\}$ to be the diagonal of any diagonalizable self-adjoint operator with the property that the largest and smallest eigenvalues have infinite multiplicity. In particular, we will prove the following theorem, which will complete the proof of part (c) of Theorem 1.2.

suff2 **Theorem 4.2.** *Let $\Lambda \subset [0, B]$ be a countable set with $0, B \in \Lambda$. Set $n_0 = n_B = \infty$, and for each $\lambda \in \Lambda \cap (0, B)$ let $n_\lambda \in \mathbb{N} \cup \{\infty\}$. If $\{d_i\}_{i \in I}$ is a sequence in $[0, B]$ such that for some (and hence all) $\alpha \in (0, B)$ we have*

$$\sum_{d_i < \alpha} d_i + \sum_{d_i \geq \alpha} (B - d_i) = \infty,$$

then there is a positive diagonalizable operator E with diagonal $\{d_i\}$, eigenvalues Λ and $m_E(\lambda) = n_\lambda$ for each $\lambda \in \Lambda$.

To prove Theorem 4.2 we need two lemmas. The first lemma will also be used in later sections, for the proof see [7, Lemmas 4.3 and 4.4].

ops **Lemma 4.3.** *Let $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$. Let $F_0, F_1 \subset I$ be two disjoint finite subsets such that $\max\{d_i : i \in I_0\} \leq \min\{d_i : i \in I_1\}$. Let $\eta_0 \geq 0$ and*

$$\eta_0 \leq \min \left\{ \sum_{i \in F_0} d_i, \sum_{i \in F_1} (B - d_i) \right\}.$$

(i) *There exists a sequence $\{\tilde{d}_i\}_{i \in I}$ in $[0, B]$ satisfying*

ops0 (4.4)
$$\tilde{d}_i = d_i \quad \text{for } i \in I \setminus (F_0 \cup F_1),$$

ops1 (4.5)
$$\tilde{d}_i \leq d_i \quad i \in F_0, \quad \text{and} \quad \tilde{d}_i \geq d_i, \quad i \in F_1,$$

ops2 (4.6)
$$\eta_0 + \sum_{i \in F_0} \tilde{d}_i = \sum_{i \in F_0} d_i \quad \text{and} \quad \eta_0 + \sum_{i \in F_1} (B - \tilde{d}_i) = \sum_{i \in F_1} (B - d_i).$$

(ii) *For any self-adjoint operator \tilde{E} on \mathcal{H} with diagonal $\{\tilde{d}_i\}_{i \in I}$, there exists an operator E on \mathcal{H} unitarily equivalent to \tilde{E} with diagonal $\{d_i\}_{i \in I}$.*

The second lemma will serve as a building block for constructing the operators in Theorem 4.2.

suff1 **Lemma 4.4.** *Let $0 < A < B < \infty$ and let $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$. If $C + D = \infty$ then there is a positive operator E with $\sigma(E) = \{0, A, B\}$, $m_E(0) = m_E(B) = \infty$, $m_E(A) = 1$, and diagonal $\{d_i\}$.*

Proof. Assume $C = \infty$. There exists $i_0 \in I_1 = \{i \in I : d_i < A\}$ such that

$$\sum_{d_i \leq d_{i_0}} d_i > A.$$

This implies that

$$\sum_{\substack{d_i \leq d_{i_0} \\ i \neq i_0}} d_i > A - d_{i_0}.$$

Let F_0 be a finite subset of $\{i \in I_1 \setminus \{i_0\} : d_i \leq d_{i_0}\}$ such that

$$\sum_{i \in F_0} d_i > A - d_{i_0}.$$

Apply Lemma 4.3 (i) with F_0 as above, $F_1 = \{i_0\}$, and $\eta_0 = A - d_{i_0}$ to obtain a sequence $\{\tilde{d}_i\}_{i \in I}$. Note that $\tilde{d}_{i_0} = A$ and since F_0 is finite

$$\sum_{i \in I_1 \setminus \{i_0\}} \tilde{d}_i = \infty.$$

Theorem 2.3 implies there is a projection Q such that BQ has diagonal $\{\tilde{d}_i\}_{i \in I \setminus \{i_0\}}$. Moreover, we have

$$\dim \ker P = \sum_{i \in I \setminus \{i_0\}} \left(1 - \frac{1}{B} \tilde{d}_i\right) = \infty = \frac{1}{B} \sum_{i \in I \setminus \{i_0\}} \tilde{d}_i = \dim \operatorname{ran} P.$$

Let P be the identity on a one-dimensional Hilbert space. The operator $\tilde{E} = BQ \oplus AP$ has diagonal $\{\tilde{d}_i\}_{i \in I}$ as well as the desired spectrum and multiplicities. Finally, by Lemma 4.3 (ii) there is an operator E , unitarily equivalent to \tilde{E} , with diagonal $\{d_i\}_{i \in I}$. This completes the proof of the theorem when $C = \infty$.

Assume $D = \infty$. Define $d'_i = B - d_i$ for each $i \in I$. We have

$$\sum_{d'_i \leq B-A} d'_i = \sum_{d_i \geq A} (B - d_i) = D = \infty.$$

By the previous argument, there is a positive operator E' with diagonal $\{d'_i\}$ and $\sigma(E') = \{0, B - A, B\}$, with 0 and B having infinite multiplicity and $B - A$ having multiplicity 1. Clearly $E = B - E'$ has the desired properties. \square

Proof of Theorem 4.2. If $\Lambda = \{0, B\}$ then Theorem 2.3 gives the desired operator. Thus, we may assume there is some $\lambda \in \Lambda$ with $0 < \lambda < B$. Set $I_1 = \{i \in I : d_i < \alpha\}$ and $I_2 = \{i : d_i \geq \alpha\}$. Partition I_1 and I_2 into (possibly empty) sets $\{I_1^\lambda\}_{\lambda \in \Lambda}$ and $\{I_2^\lambda\}_{\lambda \in \Lambda}$ respectively, such that for each $\lambda \in \Lambda$

$$\sum_{i \in I_1^\lambda} d_i + \sum_{i \in I_2^\lambda} (B - d_i) = \infty.$$

For each $\lambda \in \Lambda \cap (0, B)$ partition I_1^λ and I_2^λ into n_λ (possibly empty) sets $\{I_1^{\lambda,n}\}_{n=1}^{n_\lambda}$ and $\{I_2^{\lambda,n}\}_{n=1}^{n_\lambda}$ respectively, such that for each $n = 1, 2, \dots, n_\lambda$ we have

$$\sum_{i \in I_1^{\lambda,n}} d_i + \sum_{i \in I_2^{\lambda,n}} (B - d_i) = \infty.$$

By Lemma 4.4, for each $\lambda \in \Lambda \cap (0, B)$ and each $n = 1, 2, \dots, n_\lambda$ there is a self-adjoint operator $E_{\lambda,n}$ with diagonal $\{d_i\}_{i \in I_1^{\lambda,n} \cup I_2^{\lambda,n}}$ and $\sigma(E_{\lambda,n}) = \{0, \lambda, B\}$ with infinite multiplicity at 0 and B and multiplicity 1 at λ . Finally, set

$$E = \bigoplus_{\lambda \in \Lambda} \bigoplus_{n=1}^{n_\lambda} E_{\lambda,n},$$

and it is clear that E has the desired diagonal and eigenvalues. \square

In Theorem 4.2 the spectrum of E is the closure of Λ . To end this section we note that $C + D = \infty$ is a sufficient condition on a sequence to be the diagonal of a positive operator E with $\sigma(E) = K$ for any compact set $K \subset [0, B]$. Simply let Λ be a countable dense subset of K and apply Theorem 4.2 with any multiplicities $\{n_\lambda\}_{\lambda \in \Lambda}$. This gives us the following corollary.

suff3

Corollary 4.5. *Let $K \subset [0, B]$ be a compact set with $0, B \in K$. If $\{d_i\}_{i \in I}$ is a sequence in $[0, B]$ such that for some (and hence all) $\alpha \in (0, B)$ we have*

$$\sum_{d_i < \alpha} d_i + \sum_{d_i \geq \alpha} (B - d_i) = \infty,$$

then there is a positive diagonalizable operator E with diagonal $\{d_i\}$ and $\sigma(E) = K$.

5. OUTER EIGENVALUES WITH INFINITE MULTIPLICITY

In this section we will establish part (d) of Theorem 1.2, which is formulated in Theorem 5.1 below. Moreover, the proof of Theorem 1.1 is given.

N<infy

Theorem 5.1. *Let $0 < A < B < \infty$, let $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$ and let $N \in \mathbb{N}$. There is a positive operator E with $\sigma(E) = \{0, A, B\}$, $m_E(0) = m_E(B) = \infty$, $m_E(A) = N$, and diagonal $\{d_i\}_{i \in I}$ if and only if one of the following holds:*

- (i) $C + D = \infty$
- (ii) $C, D < \infty$, $\sum d_i = \sum (B - d_i) = \infty$, and there exists $k \in \mathbb{Z}$ such that

cdfin2

$$(5.1) \quad C - D = NA + kB$$

cdfin3

$$(5.2) \quad C \geq A(N + k).$$

Proof. First, we note that the necessity direction is immediate. Indeed, if (i) fails then we have $C, D < \infty$ and we use Theorem 4.1 to deduce (5.1) and (5.2). Moreover, $\{d_i\}$ and $\{B - d_i\}$ are not summable since both E and $B - E$ are positive operators with infinite dimensional range and finite spectrum, and thus they both have infinite trace.

Next, note that Theorem 4.2 implies that (i) is sufficient. Lastly, we assume that (ii) holds, and we must show that the desired operator exists. However, the proof is quite complicated and requires considering four distinct cases. First, we make a couple of observations.

Recall that $I_1 = \{i : d_i < A\}$ and $I_2 = \{i : d_i \geq A\}$. Since $C, D < \infty$ and $\sum d_i = \sum (B - d_i) = \infty$ it must be the case that $|I_1| = |I_2| = \infty$.

The following argument shows that it is enough to consider sequences $\{d_i\}$ with limit points at both 0 and B . Assume B is not a limit point of $\{d_i\}$. Since $D < \infty$, the set $I_2^0 := \{i \in I_2 : d_i < B\}$ is finite. Assume I_2^0 has M elements. Let $L \subset I_2 \setminus I_2^0$ be a set with $|k| + 1$ elements and define $K_2 := I_2^0 \cup L$. If we consider the sequence $\{d_i\}_{i \in I_1 \cup K_2}$, then we have

$$\begin{aligned} \sum_{i \in I_1 \cup K_2} d_i &= C + (M + |k| + 1)B - \sum_{i \in K_2} (B - d_i) \\ &= C + (M + |k| + 1)B - D = NA + (M + |k| + k + 1)B \end{aligned}$$

and

$$\sum_{\substack{i \in I_1 \cup K_2 \\ d_i < A}} d_i = C \geq (N + k)A = (N + M + |k| + k + 1 - |K_2|)A.$$

By Theorem 3.1, there is a positive operator E' with diagonal $\{d_i\}_{i \in I_1 \cup K_2}$, $\sigma(E') = \{0, A, B\}$, $m_{E'}(0) = \infty$, $m_{E'}(A) = N$ and $m_{E'}(B) = M + |k| + k + 1$. Let I be the identity operator on an infinite dimensional Hilbert space. The operator $E = E' \oplus BI$ is as desired.

If 0 is not a limit point, then we can use the above argument on the sequence $\{B - d_i\}$ to obtain an operator F with diagonal $\{B - d_i\}$ and eigenvalues 0, $B - A$ and B which have multiplicities ∞ , N and ∞ , respectively. Then $B - F$ is the desired operator. For the rest of the proof we can and will assume that both 0 and B are limit points of $\{d_i\}$.

Case 1: Assume $k \geq 0$. Since B is a limit point of $\{d_i\}$ we have $D > 0$ and thus $C = NA + kB + D > NA + kB$. There is a finite set $F_0 \subset I_1$ such that

$$\sum_{i \in F_0} d_i > NA + kB$$

Since 0 is a limit point of $\{d_i\}_{i \in I_1}$ and F_0 is finite we have $\sum_{i \in F_0} d_i < C$. Define

$$\eta_0 := \sum_{i \in F_0} d_i - NA - kB < C - NA - kB = D.$$

There is a finite set $F_1 \subset I_2$ such that

$$\sum_{i \in F_1} (B - d_i) > \eta_0.$$

The sequences $\{d_i\}_{i \in F_0}$ and $\{d_i\}_{i \in F_1}$ are in $[0, B]$, satisfy $\max\{d_i\}_{i \in F_0} \leq \min\{d_i\}_{i \in F_1}$ and

$$\eta_0 \leq \max \left\{ \sum_{i \in F_0} d_i, \sum_{i \in F_1} (B - d_i) \right\}.$$

Apply Lemma 4.3 (i) with F_0 , F_1 , and η_0 as above, to obtain a sequence $\{\tilde{d}_i\}_{i \in I}$. From (4.6) we have

$$\sum_{i \in F_0} \tilde{d}_i = \left(\sum_{i \in F_0} d_i \right) - \eta_0 = NA + kB.$$

We wish to apply Theorem 3.1 to the sequence $\{\tilde{d}_i\}_{i \in F_0}$, and this shows that (3.1) holds. From (4.5) we see that $\tilde{d}_i \leq d_i < A$ for all $i \in F_0$. From this it is clear that (3.2) also holds. We conclude that there is a positive operator \tilde{E}_0 with diagonal $\{\tilde{d}_i\}_{i \in F_0}$, $\sigma(\tilde{E}_0) = \{0, A, B\}$, $m_{\tilde{E}_0}(B) = k$, $m_{\tilde{E}_0}(A) = N$ and $m_{\tilde{E}_0}(0) = |F_0| - k - N$. Using (4.4) we have

$$\sum_{i \in I_1 \setminus F_0} \tilde{d}_i = \sum_{i \in I_1 \setminus F_0} d_i = C - \sum_{i \in F_0} d_i = D - \eta_0$$

and from (4.6) we see that

$$\sum_{i \in I_2} (B - \tilde{d}_i) = D - \eta_0.$$

By Theorem 2.3 there is a projection Q such that BQ has diagonal $\{\tilde{d}_i\}_{i \in (I_1 \setminus F_0) \cup I_2}$. Since $|I_1 \setminus F_0| = |I_2| = \infty$ we have $m_Q(1) = m_Q(0) = \infty$. Thus, the operator $\tilde{E} = \tilde{E}_0 \oplus BQ$ has the desired eigenvalues and multiplicities and diagonal $\{\tilde{d}_i\}_{i \in I}$. Finally, use the second part of Lemma 4.3 to obtain an operator E , unitarily equivalent to \tilde{E} , with diagonal $\{d_i\}_{i \in I}$. This completes the proof of the first case.

Case 2: Assume $k \leq -N$. We obtain this case by applying Case 1 to the sequence $\{B - d_i\}$, to obtain the operator E_0 with $\sigma(E_0) = \{0, B - A, B\}$, $\dim \ker(E_0) = \dim \ker(B - E_0) = \infty$ and $\dim \ker((B - A) - E_0) = N$. The operator $B - E_0$ has the desired diagonal, eigenvalues, and multiplicities.

Case 3: Assume $-N < k < 0$ and $C = A(N + k)$. Theorem 2.3 implies there is a projection P with $N + k$ dimensional range, such that AP has diagonal $\{d_i\}_{i \in I_1}$. Since $|I_1| = \infty$ we also see that P has infinite dimensional kernel.

Next, note that

$$\sum_{i \in I_2} (B - d_i) = D = C - NA - kB = NA + kA - NA - kB = -k(B - A).$$

Theorem 2.3 implies that there is a projection Q with $-k$ dimensional range, such that $(B - A)Q$ has diagonal $\{B - d_i\}_{i \in I_2}$. Since $|I_2| = \infty$ we see that Q has infinite dimensional kernel. The operator $E = AP \oplus (BI - (B - A)Q)$ has the desired diagonal, eigenvalues, and multiplicities.

Case 4: Assume $-N < k < 0$ and $C > A(N + k)$. Set $\eta_0 := C - (N + k)A < C$. There is a finite set $F_0 \subset I_1$ such that

$$\sum_{i \in F_0} d_i > \eta_0.$$

Next, note that

$$\eta_0 = C - (N + k)A = NA + kB + D - NA - kA = D + k(B - A) < D.$$

Thus, there is a finite set $F_1 \subset I_2$ such that

$$\sum_{i \in F_1} (B - d_i) > \eta_0.$$

Apply Lemma 4.3 (i) with F_0, F_1 , and η_0 as above, to obtain a sequence $\{\tilde{d}_i\}_{i \in I}$. Using (4.6) we have

$$\sum_{i \in I_1} \tilde{d}_i = \sum_{i \in I_1 \setminus F_0} d_i + \sum_{i \in F_0} \tilde{d}_i = \sum_{i \in I_1 \setminus F_0} d_i + \sum_{i \in F_0} d_i - \eta_0 = C - \eta_0 = (N + k)A$$

and

$$\sum_{i \in I_2} (B - \tilde{d}_i) = \sum_{i \in I_2 \setminus F_1} (B - d_i) + \sum_{i \in F_1} (B - d_i) - \eta_0 = D - \eta_0 = -k(B - A).$$

Thus, the sequence $\{\tilde{d}_i\}_{i \in I}$ satisfies the conditions of Case 3, so there is an operator \tilde{E} with the desired eigenvalues and multiplicities but with diagonal $\{\tilde{d}_i\}_{i \in I}$. The second part of Lemma 4.3 implies there is an operator E , unitarily equivalent to \tilde{E} , but with diagonal $\{d_i\}_{i \in I}$. This completes the final case. \square

We are now in a position to prove Theorem 1.1. In fact we will prove the following more general theorem.

3ptg

Theorem 5.2. *Let $0 < A < B < \infty$ and let $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$. If there is a positive operator E with diagonal $\{d_i\}_{i \in I}$ and $\sigma(E) = \{0, A, B\}$ then one following holds:*

- (i) $C = \infty$,
- (ii) $D = \infty$,
- (iii) $C, D < \infty$ and there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that (1.2) and (1.3) hold.

Conversely, if $\sum d_i = \sum(B - d_i) = \infty$ and one of (i), (ii), or (iii) holds, then there is a positive operator E with diagonal $\{d_i\}_{i \in I}$ and $\sigma(E) = \{0, A, B\}$.

Proof. First, assume that E is a positive operator with spectrum $\{0, A, B\}$ and diagonal $\{d_i\}$. If either $C = \infty$ or $D = \infty$ then we are done since this is exactly (i) or (ii). If $C, D < \infty$ then Theorem 5.1 shows that (1.2) and (1.3) hold and thus (iii) holds.

Next, assume $\{d_i\}$ is a sequence in $[0, B]$. If (i) or (ii) holds then Theorem 4.2 shows that there is a positive operator E with spectrum $\{0, A, B\}$ and diagonal $\{d_i\}$. Finally, if (iii) holds and $\sum d_i = \sum(B - d_i) = \infty$ then Theorem 5.1 shows that there is a positive operator E with spectrum $\{0, A, B\}$ and diagonal $\{d_i\}$. \square

rmk1 *Remark 5.1.* In Theorem 1.1 (and Theorem 5.2) the assumption that $\sum d_i = \sum(B - d_i) = \infty$ is necessary. Consider the sequence $\{A, 0, 0, \dots\}$. This is clearly not the diagonal of any operator with spectrum $\{0, A, B\}$ since the operator would be trace class with trace equal to A , and thus $B > A$ cannot be an eigenvalue. However, we have $C = 0$ and $D = B - A$ so that (1.2) and (1.3) hold with $N = 1$ and $k = -1$.

rmk2 *Remark 5.2.* There exist two non-unitarily equivalent operators with three point spectrum and the same diagonal. Let $0 < A < B$ and let I_n be the identity operator of an n dimensional Hilbert space. From Theorem 2.3, there is a projection P with infinite dimensional kernel and range such that the diagonal of BP consists of a countable infinite sequence of A 's. The operator $BP \oplus AI_n$ has a diagonal consisting of a countable number of A 's, however the multiplicity of the eigenvalue A is n .

6. OUTER EIGENVALUE WITH FINITE MULTIPLICITY

In the last two remaining cases ((e) and (f)) of Theorem 1.2 we consider operators with finite dimensional kernel. In these cases, where there is an “outer” eigenvalue with finite multiplicity, we have the following necessary condition.

nec6 **Theorem 6.1.** Let $0 < A < B < \infty$ and let E be a positive operator on a Hilbert space \mathcal{H} with $\sigma(E) = \{0, A, B\}$ and $m_E(0) < \infty$. Let $\{e_i\}_{i \in I}$ be an orthonormal basis for \mathcal{H} and set $d_i = \langle Ee_i, e_i \rangle$. We have

nec6.1 (6.1)
$$\sum_{d_i < A} (A - d_i) \leq Am_E(0).$$

Proof. There exist mutually orthogonal projections P and Q such that $E = AP + BQ$. Note that $I - P - Q$ is a finite rank projection and thus has finite trace equal to $m_E(0)$. Set $J_1 = \{i \in I : d_i < A\}$. Then

$$\begin{aligned} \sum_{i \in J_1} (A - d_i) &= \sum_{i \in J_1} (A - A\langle Pe_i, e_i \rangle - B\langle Qe_i, e_i \rangle) \leq \sum_{i \in J_1} (A - A\langle Pe_i, e_i \rangle - A\langle Qe_i, e_i \rangle) \\ &= A \left(\sum_{i \in J_1} (1 - \langle Pe_i, e_i \rangle - \langle Qe_i, e_i \rangle) \right) \leq A \left(\sum_{i \in I} (1 - \langle Pe_i, e_i \rangle - \langle Qe_i, e_i \rangle) \right) \\ &= Am_E(0). \end{aligned}$$

\square

Next, we look at two examples which demonstrate that for operators with finite dimensional kernel the constants C and D do not capture enough information about a sequence in order to tell if it is the diagonal of an operator of the specified type.

Example 1. Consider the sequence $\{d_i\}$ consisting of $\{1 - i^{-1}\}_{i=1}^{\infty}$ and a countable infinite number of 2's. If $A = 1$ and $B = 2$ then we have $C = \infty$ and $D = 0$. By Theorem 6.1 this is not the diagonal of any positive operator E with $\sigma(E) = \{0, 1, 2\}$ and finite dimensional kernel, since

$$\sum_{d_i < A} (A - d_i) = \sum_{i=1}^{\infty} \frac{1}{i} = \infty.$$

Example 2. Consider the sequence $\{c_i\}$ consisting of $\{1 - 2^{-i}\}_{i=1}^{\infty}$ and a countable infinite number of 2's. If $A = 1$ and $B = 2$ then we have $C = \infty$ and $D = 0$. By Theorem 2.3 there is a projection P with diagonal $\{1 - 2^{-i}\}_{i=1}^{\infty}$ and finite dimensional kernel. Let I be the identity operator on an infinite dimensional Hilbert space and set $E = P \oplus 2I$. This operator has diagonal $\{c_i\}$, spectrum $\{0, 1, 2\}$ and finite dimensional kernel. Note that $\{c_i\}$ and $\{d_i\}$ have the same values for C and D , but only $\{c_i\}$ is the diagonal of an operator with spectrum $\{0, 1, 2\}$ and finite dimensional kernel.

Instead of C and D we will use the following terminology from Theorem 1.2 in the rest of the section:

$$J_1 = \{i : d_i < A\}, \quad J_2 = \left\{i : d_i \in \left[A, \frac{A+B}{2}\right)\right\}, \quad J_3 = \left\{i : d_i \geq \frac{A+B}{2}\right\}$$

$$C_1 = \sum_{i \in J_1} (A - d_i), \quad C_2 = \sum_{i \in J_2} (d_i - A), \quad C_3 = \sum_{i \in J_3} (B - d_i)$$

Note that for symmetry we will use the notation J_1 instead of I_1 , though they denote the same set.

The next theorem shows the necessity of the conditions in part (e) of Theorem 1.2.

nec7 **Theorem 6.2.** *Let $0 < A < B < \infty$ and let E be a positive operator on a Hilbert space \mathcal{H} with $\sigma(E) = \{0, A, B\}$, $m_E(0) < \infty$, and $m_E(A) = m_E(B) = \infty$. Let $\{e_i\}_{i \in I}$ be an orthonormal basis for \mathcal{H} and set $d_i = \langle E e_i, e_i \rangle$. If $C_2, C_3 < \infty$, then $C_1 < \infty$, $|J_1 \cup J_2| = |J_3| = \infty$, and there exist $n, k \in \mathbb{Z}$ such that $n + k = m_E(0)$,*

nec7.1 (6.2)
$$C_1 - C_2 + C_3 = nA + kB,$$

and

nec7.2 (6.3)
$$C_1 \leq A(n + k).$$

Proof. There exist mutually orthogonal projections P and Q such that $E = AP + BQ$. Define $p_i = \langle P e_i, e_i \rangle$ and $q_i = \langle Q e_i, e_i \rangle$ for each $i \in I$. Since $m_E(0) < \infty$, Theorem 6.1 implies that $C_1 \leq A m_E(0) < \infty$. Next, we note that

nec7.3 (6.4)
$$\sum_{i \in I} (1 - p_i - q_i) = m_{P+Q}(0) = m_E(0) < \infty.$$

Using (6.4) we have

$$\begin{aligned} \sum_{i \in J_1 \cup J_2} q_i &= \frac{1}{B-A} \left(\sum_{i \in J_1 \cup J_2} (A - Ap_i - Aq_i) - \sum_{i \in J_1} (A - Ap_i - Bq_i) + \sum_{i \in J_2} (Bq_i + Ap_i - A) \right) \\ &= \frac{1}{B-A} \left(\sum_{i \in J_1 \cup J_2} (A - Ap_i - Aq_i) - C_1 + C_2 \right) \leq \frac{Am_E(0) - C_1 + C_2}{B-A} < \infty. \end{aligned}$$

Together with (6.4) this also shows that $\sum_{i \in J_1 \cup J_2} (1 - p_i) < \infty$. A similar calculation shows that

$$\sum_{i \in J_3} (1 - q_i), \sum_{i \in J_3} p_i < \infty.$$

By Theorem 2.3 there exist $n, k \in \mathbb{Z}$ such that

$$\begin{aligned} n &= \sum_{i \in J_1 \cup J_2} (1 - p_i) - \sum_{i \in J_3} p_i \\ k &= \sum_{i \in J_3} (1 - q_i) - \sum_{i \in J_1 \cup J_2} q_i. \end{aligned} \tag{eq2} \tag{6.5}$$

Now, we calculate

$$\begin{aligned} C_1 - C_2 + C_3 &= \sum_{i \in J_1} (A - Ap_i - Bq_i) - \sum_{i \in J_2} (Ap_i + Bq_i - A) + \sum_{i \in J_3} (B - Ap_i - Bq_i) \\ &= A \sum_{i \in J_1 \cup J_2} (1 - p_i) - A \sum_{i \in J_3} p_i + B \sum_{i \in J_3} (1 - q_i) - B \sum_{i \in J_1 \cup J_2} q_i \\ &= nA + kB, \end{aligned}$$

which shows (6.2) holds.

From (6.5) we have

$$n + k = \sum_{i \in I} (1 - p_i - q_i) = m_E(0).$$

Theorem 6.1 shows $C_1 \leq Am_E(0) = A(n + k)$, which is (6.3).

Note that

$$\sum_{i \in I} p_i = \dim \operatorname{ran} P = m_E(A) = \infty.$$

Since $\sum_{i \in J_3} p_i < \infty$, it must be the case that $\sum_{i \in J_1 \cup J_2} p_i = \infty$ and thus $|J_1 \cup J_2| = \infty$. Similarly, since Q has infinite dimensional range, we have $\sum_{i \in I} q_i = \infty$. Since $\sum_{i \in J_1 \cup J_2} q_i < \infty$ it must be the case that $\sum_{i \in J_3} q_i = \infty$, and thus $|J_3| = \infty$. \square

The next theorem shows that the conditions in part (e) of Theorem 1.2 are sufficient to construct the desired operator. We state it in a slightly more general form for use in the proof of part (f) later in this section.

suff4 **Theorem 6.3.** *Let $0 < A < B < \infty$, let $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$, and let $Z \in \mathbb{N}$. If $|J_1 \cup J_2| = \infty$, $C_1 \leq AZ$, and either of the following holds:*

- (i) $C_2 + C_3 = \infty$
- (ii) $C_2, C_3 < \infty$ and there exists $n, k \in \mathbb{Z}$ such that $Z = n + k$ and

$$\text{trace} \tag{6.6} \quad C_1 - C_2 + C_3 = nA + kB,$$

then there is a positive operator E with $\sigma(E) = \{0, A, B\}$, $m_E(0) = Z$, $m_E(A) = \infty$, and diagonal $\{d_i\}$. Moreover, if (i) holds then $m_E(B) = \infty$, and if (ii) holds then $m_E(B) = |J_3| - k$.

Proof. Set

$$\eta = AZ - C_1.$$

Case 1: Assume

$$\sum_{i \in J_1} d_i, \sum_{i \in J_2 \cup J_3} (B - d_i) > \eta.$$

There are finite subsets $F_0 \subset J_1$ and $F_1 \subset J_2 \cup J_3$ such that

$$\eta \leq \min \left\{ \sum_{i \in F_0} d_i, \sum_{i \in F_1} (B - d_i) \right\}.$$

We can apply Lemma 4.3 (i) with F_0 and F_1 as above, and $\eta_0 = \eta$, to obtain $\{\tilde{d}_i\}_{i \in I}$. From (4.6) we have

$$\begin{aligned} \sum_{i \in J_1} (A - \tilde{d}_i) &= \sum_{i \in F_0} (A - \tilde{d}_i) + \sum_{i \in J_1 \setminus F_0} (A - d_i) = |F_0|A - \sum_{i \in F_0} \tilde{d}_i + \sum_{i \in J_1 \setminus F_0} (A - d_i) \\ &= |F_0|A + \eta - \sum_{i \in F_0} d_i + \sum_{i \in J_1 \setminus F_0} (A - d_i) = \eta + \sum_{i \in J_1} (A - d_i) = \eta + C_1 = AZ. \end{aligned}$$

Theorem 2.3 implies there is a projection P with Z dimensional kernel such that AP has diagonal $\{\tilde{d}_i\}_{i \in J_1}$. It is clear that if $|J_1| = \infty$ then $m_P(1) = \infty$.

If (i) holds, that is $C_2 + C_3 = \infty$, then Theorem 2.3 implies there is a projection Q_1 such that $(B - A)Q_1$ has diagonal $\{\tilde{d}_i - A\}_{i \in J_2 \cup J_3}$. Since

$$\sum_{i \in J_2 \cup J_3} (\tilde{d}_i - A) = \sum_{i \in J_2 \cup J_3} ((B - A) - (\tilde{d}_i - A)) = \infty,$$

we also see that $m_{Q_1}(0) = m_{Q_1}(1) = \infty$. Set $\tilde{E} = AP \oplus ((B - A)Q_1 + AI)$. It is clear that $m_{\tilde{E}}(0) = Z$, $\sigma(\tilde{E}) = \{0, A, B\}$, and $m_{\tilde{E}}(A) = m_{\tilde{E}}(B) = \infty$. By the second part of Lemma 4.3 there is an operator E , unitarily equivalent to \tilde{E} , with diagonal $\{d_i\}_{i \in I}$.

If (ii) holds, then using (6.6) we have

$$\begin{aligned} \sum_{i \in J_2} (\tilde{d}_i - A) - \sum_{i \in J_3} (B - \tilde{d}_i) &= \eta + \sum_{i \in J_2} (d_i - A) - \sum_{i \in J_3} (B - d_i) = \eta + C_2 - C_3 \\ &= AZ - C_1 + C_1 - An - Bk = -k(B - A). \end{aligned}$$

Theorem 2.3 implies there is a projection Q_2 such that $(B - A)Q_2$ has diagonal $\{\tilde{d}_i - A\}_{i \in J_2 \cup J_3}$. The operator $\tilde{E} = AP \oplus ((B - A)Q_2 + AI)$ has diagonal $\{\tilde{d}_i\}_{i \in I}$, and it is clear that $m_{\tilde{E}}(0) = Z$ and $\sigma(\tilde{E}) = \{0, A, B\}$. Note that if $|J_2| = \infty$ then $m_{Q_2}(0) = \infty$ and we already noted that $|J_1| = \infty$ implies $m_P(1) = \infty$; in either case $m_{\tilde{E}}(A) = \infty$. If $|J_3| = \infty$ we have $m_{Q_2}(1) = \infty$ and thus $m_{\tilde{E}}(B) = \infty$. If $|J_3| < \infty$ then we have

$$\sum_{i \in J_2 \cup J_3} (\tilde{d}_i - A) = \sum_{i \in J_2} (\tilde{d}_i - A) - \sum_{i \in J_3} (B - \tilde{d}_i) + |J_3|(B - A) = (|J_3| - k)(B - A),$$

which implies $m_{Q_2}(1) = |J_3| - k$, and thus $m_{\tilde{E}}(B) = |J_3| - k$. By Lemma 4.3 (ii), there is an operator E , unitarily equivalent to \tilde{E} , with diagonal $\{d_i\}$. This completes the proof of Case 1.

Case 2: Assume

$$\sum_{i \in J_1} d_i \leq \eta.$$

This implies J_1 is a finite set and that $|J_1| \leq Z$. Since $|J_1 \cup J_2| = \infty$ this implies $|J_2| = \infty$, and thus

$$\sum_{i \in J_2} (B - d_i) = \infty.$$

Let $L, F_1 \subset J_2 \cup J_3$ be disjoint finite sets which satisfy three conditions:

$$\sum_{i \in F_1} (B - d_i) > BZ,$$

$|L| = Z - |J_1|$, and $\max\{d_i\}_{i \in L} \leq \min\{d_i\}_{i \in F_1}$. Set $F_0 = J_1 \cup L$. Apply Lemma 4.3 (i) with F_0 and F_1 as already defined, and

$$\eta_0 = \sum_{i \in F_0} d_i < BZ$$

to obtain the sequence $\{\tilde{d}_i\}_{i \in I}$. The choice of η_0 implies that $\{\tilde{d}_i\}_{i \in F_0}$ is a sequence of Z zeroes.

If (i) holds, then we have

$$\sum_{i \in J_2} (\tilde{d}_i - A) + \sum_{i \in J_3} (B - \tilde{d}_i) = \infty.$$

Theorem 2.3 implies that there is a projection Q_1 such that $(B - A)Q_1$ has diagonal $\{\tilde{d}_i - A\}_{i \in J_2 \cup J_3}$ and $m_{Q_1}(0) = m_{Q_1}(1) = \infty$. Let $\mathbf{0}_Z$ be the zero operator on a Z dimensional Hilbert space, and set $\tilde{E} = \mathbf{0}_Z \oplus ((B - A)Q_1 + AI)$. It is clear that \tilde{E} has diagonal $\{\tilde{d}_i\}$, $m_{\tilde{E}}(0) = Z$, $\sigma(\tilde{E}) = \{0, A, B\}$, and $m_{\tilde{E}}(A) = m_{\tilde{E}}(B) = \infty$. By Lemma 4.3 (ii), there is an operator E , unitarily equivalent to \tilde{E} , with diagonal $\{d_i\}_{i \in I}$.

If (ii) holds then by (6.6) we have

$$\begin{aligned} \sum_{i \in J_2 \setminus L} (\tilde{d}_i - A) - \sum_{i \in J_3} (B - \tilde{d}_i) &= \eta_0 + \sum_{i \in J_2 \setminus L} (d_i - A) - \sum_{i \in J_3} (B - d_i) \\ &= \sum_{i \in J_1} d_i + \sum_{i \in L} d_i + \sum_{i \in J_2 \setminus L} (d_i - A) - C_3 \\ &= -C_1 + C_2 - C_3 + (|J_1| + |L|)A \\ &= -nA - kB + ZA = -k(B - A). \end{aligned}$$

Theorem 2.3 implies there is a projection Q_2 such that $(B - A)Q_2$ has diagonal $\{\tilde{d}_i - A\}_{i \in (J_2 \cup J_3) \setminus L}$. Since J_2 is infinite we have $m_{Q_2}(0) = \infty$. If J_3 is infinite then we also have $m_{Q_2}(1) = \infty$. If $|J_3| < \infty$ then

$$\sum_{i \in (J_2 \cup J_3) \setminus L} (\tilde{d}_i - A) = \sum_{i \in J_2 \setminus L} (\tilde{d}_i - A) - \sum_{i \in J_3} (B - \tilde{d}_i) + |J_3|(B - A) = (|J_3| - k)(B - A),$$

which implies $m_{Q_2}(1) = |J_3| - k$. The operator $\tilde{E} = \mathbf{0}_Z \oplus ((B - A)Q_2 + AI)$ has the desired eigenvalues and multiplicities and diagonal $\{\tilde{d}_i\}$. Lemma 4.3 (ii) implies there is an operator E , unitarily equivalent to \tilde{E} , with diagonal $\{d_i\}$. This completes the proof of the second case.

Case 3: Assume

$$\sum_{i \in J_2 \cup J_3} (B - d_i) \leq \eta.$$

This implies J_2 is finite, since $d_i < (B + A)/2$ for all $i \in J_2$. By hypothesis $|J_1 \cup J_2| = \infty$, and thus J_1 must be infinite. Moreover, A is a limit point of $\{d_i\}_{i \in J_1}$, since $\sum_{i \in J_1} (A - d_i) < \infty$ and $d_i < A$ for all $i \in J_1$. There is some $N_0 \in \mathbb{N}$ such that

$$(B - A)N_0 > \eta.$$

Choose $\alpha \in (0, A)$ such that

$$\sum_{d_i < \alpha} d_i > AN_0.$$

Set $F_0 = \{i \in J_1 : d_i < \alpha\}$, and note that it is finite since $C_1 < \infty$. Since A is a limit point of $\{d_i\}_{i \in J_1}$, we can find a set $F_1 \subset \{i \in J_1 : d_i \geq \alpha\}$ with N_0 elements, and clearly

$$\sum_{i \in F_1} (A - d_i) < AN_0.$$

Applying Lemma 4.3 (i) on the interval $[0, A]$, with F_0 and F_1 as above, and

$$\eta_0 = \sum_{i \in F_1} (A - d_i),$$

we obtain a sequence $\{\tilde{d}_i\}_{i \in I}$. Using (4.6) we see that $\tilde{d}_i = A$ for each $i \in F_1$. We also have

$$\sum_{i \in F_0} (A - \tilde{d}_i) = |F_0|A - \sum_{i \in F_0} \tilde{d}_i = |J_1|A - \sum_{i \in F_0} d_i - \sum_{i \in F_1} (A - d_i) = \sum_{i \in F_0 \cup F_1} (A - d_i).$$

Define the sets

$$\tilde{J}_1 = \{i : \tilde{d}_i < A\}, \quad \tilde{J}_2 = \left\{i : \tilde{d}_i \in \left[A, \frac{A+B}{2}\right)\right\}, \quad \tilde{J}_3 = \left\{i : \tilde{d}_i \geq \frac{A+B}{2}\right\}.$$

We have

$$\begin{aligned} \sum_{i \in \tilde{J}_1} (A - \tilde{d}_i) &= \sum_{i \in J_1 \setminus (F_0 \cup F_1)} (A - d_i) + \sum_{i \in F_0} (A - \tilde{d}_i) = \sum_{i \in J_1 \setminus (F_0 \cup F_1)} (A - d_i) + \sum_{i \in F_0 \cup F_1} (A - d_i) \\ &= \sum_{i \in J_1} (A - d_i) = C_1. \end{aligned}$$

Since $\tilde{d}_i = A$ for all $i \in F_1$ we have

$$\sum_{i \in \tilde{J}_2} (\tilde{d}_i - A) = \sum_{i \in J_2} (d_i - A) + \sum_{i \in F_1} (\tilde{d}_i - A) = \sum_{i \in J_2} (d_i - A) = C_2.$$

Lastly, $\tilde{d}_i = d_i$ for all $i \in J_3$, and thus

$$\sum_{i \in \tilde{J}_3} (B - \tilde{d}_i) = C_3.$$

However,

$$\sum_{i \in \tilde{J}_2 \cup \tilde{J}_3} (B - \tilde{d}_i) = \sum_{i \in J_2 \cup J_3} (B - d_i) + (B - A)N_0 > \eta.$$

This implies that $\{\tilde{d}_i\}_{i \in I}$ satisfies the conditions of Case 1, and thus there is an operator \tilde{E} with the desired eigenvalues and multiplicities and diagonal $\{\tilde{d}_i\}_{i \in I}$. By Lemma 4.3 (ii), there is an operator E , unitarily equivalent to \tilde{E} , with diagonal $\{d_i\}_{i \in I}$. This completes the proof of this case and the proof of the theorem. \square

As a corollary of Theorems 6.2 and 6.3 we deduce part (f) of Theorem 1.2. This will complete the proof of Theorem 1.2.

part(f)

Corollary 6.4. *Let $0 < A < B < \infty$, let $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$, and let $Z, K \in \mathbb{N}$. There exists a positive operator E with $\sigma(E) = \{0, A, B\}$, $m_E(0) = Z$, $m_E(A) = \infty$, $m_E(B) = K$ and diagonal $\{d_i\}$ if and only if $|I| = \infty$, $C_1 \leq ZA$ and*

trace4

$$(6.7) \quad \sum_{i \in I} (d_i - A) = K(B - A) - ZA.$$

Proof. First, assume that $|I| = \infty$, $C_1 \leq ZA$ and (6.7) holds. It is clear that $|J_3| < \infty$ and thus $|J_1 \cup J_2| = |I \setminus J_3| = \infty$. We have

$$C_1 - C_2 + C_3 = - \sum_{i \in I} (d_i - A) + |J_3|(B - A) = (Z + K - |J_3|)A + (|J_3| - K)B.$$

By Theorem 6.3 the desired operator exists.

Next, assume the operator E exists. Note that $E - A$ is a finite rank operator, and thus it is of trace class with trace

$$\sum_{i \in I} (d_i - A) = K(B - A) - AZ.$$

By Theorem 6.1 we have $C_1 \leq ZA$. Since $m_E(A) = \infty$, the operator E is acting on an infinite dimensional Hilbert space, thus $|I| = \infty$. \square

7. EXAMPLES

To demonstrate the use of Theorem 1.1 we will consider the following problem: Given a sequence $\{d_i\}$ in $[0, 1]$, for what values of A is there a positive operator E with $\sigma(E) = \{0, A, 1\}$ and diagonal $\{d_i\}$? First, we will prove the following general theorem.

exthm

Theorem 7.1. *Let $\{d_i\}_{i \in \mathbb{N}}$ be a sequence in $[0, 1]$ and set*

$$\mathcal{A} = \{A \in (0, 1) : \exists E \geq 0 \text{ with } \sigma(E) = \{0, A, 1\} \text{ and diagonal } \{d_i\}\}.$$

Either $\mathcal{A} = (0, 1)$ or \mathcal{A} is a finite (possibly empty) set.

Proof. For each $A \in (0, 1)$ define

$$C(A) = \sum_{d_i < A} d_i \quad \text{and} \quad D(A) = \sum_{d_i \geq A} (1 - d_i).$$

Note that if $C(A) + D(A) = \infty$ for some $A \in (0, 1)$ then $C(A) + D(A) = \infty$ for all $A \in (0, 1)$. By Theorem 5.2 we have $\mathcal{A} = (0, 1)$. Thus, we will assume $C(A), D(A) < \infty$ for all $A \in (0, 1)$.

First, we wish to show that $\sup \mathcal{A} < 1$. Assume to the contrary that $\sup \mathcal{A} = 1$. Note that there exists $\eta \in [0, 1)$ such that $\eta = C(A) - D(A) - \lfloor C(A) - D(A) \rfloor$ for all $A \in (0, 1)$. Thus, for each $A \in (0, 1)$ there exists $m(A) \in \mathbb{Z}$ such that

$$C(A) - D(A) = m(A) + \eta.$$

By Theorem 5.2, for each $A \in \mathcal{A}$ there exists $N(A) \in \mathbb{N}$ and $k(A) \in \mathbb{Z}$ such that

$$\boxed{\text{exthm5}} \quad (7.1) \quad m(A) + \eta = C(A) - D(A) = N(A)A + k(A) \quad \text{and} \quad C(A) \geq (N(A) + k(A))A.$$

Using (7.1) we have

$$\boxed{\text{exthm4}} \quad (7.2) \quad m(A) + \eta = N(A)A + k(A) < N(A) + k(A) \leq \frac{C(A)}{A}.$$

Since $\eta \geq 0$ and $m(A), N(A), k(A) \in \mathbb{Z}$, we can also see

$$\boxed{\text{exthm3}} \quad (7.3) \quad m(A) + 1 \leq N(A) + k(A).$$

Thus, for each $A \in \mathcal{A}$ we must have

$$\boxed{\text{exthm1}} \quad (7.4) \quad A(m(A) + 1) \leq C(A).$$

Next, note that for $A, A' \in \mathcal{A}$ with $A' > A$ we have

$$\begin{aligned} m(A') - m(A) &= C(A') - C(A) + D(A) - D(A') = \sum_{A \leq d_i < A'} d_i + \sum_{A \leq d_i < A'} (1 - d_i) \\ &= |\{i \in \mathbb{N} : A \leq d_i < A'\}|. \end{aligned}$$

Using this gives

$$\boxed{\text{exthm2}} \quad (7.5) \quad C(A') - C(A) = \sum_{d_i < A'} d_i - \sum_{d_i < A} d_i = \sum_{A \leq d_i < A'} d_i < A'(m(A') - m(A)).$$

Putting together (7.4) and (7.5) we have

$$A'(m(A') + 1) - C(A) \leq C(A') - C(A) < A'(m(A') - m(A)).$$

Rearranging this inequality gives

$$A'(m(A) + 1) < C(A).$$

Since $\sup \mathcal{A} = 1$ we can let $A' \rightarrow 1$ and we have

$$m(A) + 1 \leq C(A).$$

Finally, since $D(A) \rightarrow 0$ as $A \rightarrow 1$, for large enough A we have $D(A) < 1 - \eta$ and thus

$$C(A) < C(A) - D(A) - \eta + 1 = m(A) + 1$$

which gives a contradiction, and shows that $A_{\sup} := \sup \mathcal{A} < 1$. A symmetric argument shows that $A_{\inf} := \inf \mathcal{A} > 0$.

Since $C(A)$ and $m(A)$ are nondecreasing as $A \rightarrow 1$, for each $A \in \mathcal{A}$ we have $C(A_{\inf}) \leq C(A) \leq C(A_{\sup})$ and $m(A_{\inf}) \leq m(A) \leq m(A_{\sup})$. Using (7.2) and (7.3), for $A \in \mathcal{A}$ we have

$$m(A_{\inf}) + 1 \leq m(A) + 1 \leq N(A) + k(A) \leq \frac{C(A)}{A} \leq \frac{C(A_{\sup})}{A_{\inf}}.$$

This shows that $\{N(A) + k(A) : A \in \mathcal{A}\}$ and $\{m(A) : A \in \mathcal{A}\}$ are finite sets of integers. Next, we note that for $A \in \mathcal{A}$ we have

$$N(A)A_{\sup} \geq N(A)A = m(A) + \eta - k(A) \geq m(A_{\inf}) + \eta + N(A) - \frac{C(A_{\sup})}{A_{\inf}}.$$

Rearranging this inequality gives

$$N(A) \leq \frac{\frac{C(A_{\sup})}{A_{\inf}} - m(A_{\inf}) - \eta}{1 - A_{\sup}},$$

which implies that $\{N(A) : A \in \mathcal{A}\} \subset \mathbb{N}$ is finite. Since $\{N(A) + k(A) : A \in \mathcal{A}\}$ is finite, we also see that $\{k(A) : A \in \mathcal{A}\}$ is finite. Finally, we note that for $A \in \mathcal{A}$ we have

$$A = \frac{m(A) + \eta - k(A)}{N(A)},$$

which clearly implies that \mathcal{A} is finite. □

Next, we will explicitly find the set \mathcal{A} from Theorem 7.1 for two particular sequences $\{d_i\}$.

Example 3. Let $\beta \in (0, 1/2)$ and define the sequence $\{d_i\}_{i \in \mathbb{Z} \setminus \{0\}}$ by

$$d_i = \begin{cases} 1 - \beta^i & i > 0 \\ \beta^{-i} & i < 0. \end{cases}$$

Define the set

$$\mathcal{A}_\beta = \{A \in (0, 1) : \exists E \geq 0 \text{ with } \sigma(E) = \{0, A, 1\} \text{ and diagonal } \{d_i\}\}.$$

We will show that

$$\mathcal{A}_\beta = \begin{cases} \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\} & \frac{-1+\sqrt{13}}{6} \leq \beta < 1/2 \\ \{\frac{1}{2}\} & 1/3 \leq \beta < \frac{-1+\sqrt{13}}{6} \\ \emptyset & 0 < \beta < 1/3. \end{cases}$$

First, assume $A \in \mathcal{A}_\beta \cap (\beta, 1 - \beta]$, and thus

$$C = \sum_{d_i < A} d_i = \sum_{i=1}^{\infty} \beta^i = \frac{\beta}{1 - \beta} \quad \text{and} \quad D = \sum_{d_i \geq A} (1 - d_i) = \sum_{i=1}^{\infty} \beta^i = \frac{\beta}{1 - \beta}.$$

From Theorem 5.2 there exists $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$\boxed{\text{ex1}} \quad (7.6) \quad 0 = C - D = NA + k$$

$$\boxed{\text{ex2}} \quad (7.7) \quad \frac{\beta}{1 - \beta} = C \geq (N + k)A.$$

Using (7.6) and $A \leq 1 - \beta$ we have

$$0 < \beta N = NA + k + \beta N \leq N(1 - \beta) + k + \beta N = N + k,$$

and thus $N + k > 0$. Now, we use (7.7), $\beta < A$, then $\beta < 1/2$ to see

$$N + k < (N + k) \frac{A}{\beta} \leq \beta^{-1} \frac{\beta}{1 - \beta} = \frac{1}{1 - \beta} < 2.$$

Since $N + k \in \mathbb{Z}$ we see that $N + k = 1$. Solving for A in (7.6) we have

$$A = \frac{-k}{N} = \frac{N-1}{N} = 1 - \frac{1}{N}.$$

Since $\{d_i\}$ is symmetric about $1/2$, if $A \in \mathcal{A}_\beta$ then $1 - A \in \mathcal{A}_\beta$. And since $A = 1 - 1/N$ for some $N \in \mathbb{N}$, the only possible value of N is 2. For $N = 2$ we have $k = -1$ and $A = 1/2$, which satisfy (1.2) and (1.3) if and only if $\beta \geq 1/3$. Thus, $\mathcal{A}_\beta \cap (\beta, 1 - \beta] = \{1/2\}$ for $\beta \geq 1/3$ and $\mathcal{A}_\beta \cap (\beta, 1 - \beta] = \emptyset$ for $\beta < 1/3$.

Next, assume $A \in \mathcal{A}_\beta \cap (1 - \beta^m, 1 - \beta^{m+1}]$ for some $m \in \mathbb{N}$. We have

$$C = \frac{\beta}{1 - \beta} + \sum_{i=1}^m (1 - \beta^i) = m + \frac{\beta^{m+1}}{1 - \beta} \quad \text{and} \quad D = \sum_{i=m+1}^{\infty} \beta^i = \frac{\beta^{m+1}}{1 - \beta}.$$

By Theorem 5.2 there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$\boxed{\text{ex3}} \quad (7.8) \quad m = C - D = NA + k$$

$$\boxed{\text{ex4}} \quad (7.9) \quad m + \frac{\beta^{m+1}}{1 - \beta} = C \geq (N + k)A.$$

Using (7.8) and $A \leq 1 - \beta^{m+1}$ we have

$$\boxed{\text{ex5}} \quad (7.10) \quad m < m + N\beta^{m+1} = NA + k + N\beta^{m+1} \leq N(1 - \beta^{m+1}) + k + N\beta^{m+1} = N + k.$$

Using (7.9) and $A > 1 - \beta^m$ we have

$$m + \frac{\beta^{m+1}}{1 - \beta} \geq (N + k)A > (N + k)(1 - \beta^m).$$

Rearranging, and using $\beta < 1/2$ we have

$$\boxed{\text{ex6}} \quad (7.11) \quad N + k < \left(m + \frac{\beta^{m+1}}{1 - \beta}\right) \frac{1}{1 - \beta^m} < \left(m + \frac{1}{2^m}\right) \frac{2^m}{2^m - 1} = m + \frac{1 + m}{2^m - 1}.$$

A simple calculation shows that $\frac{1+m}{2^m-1} \leq 1$ for all $m \geq 2$. Combining this with (7.10) shows that $m < N + k < m + 1$ for $m \geq 2$. Since $N + k \in \mathbb{Z}$ this shows that $\mathcal{A}_\beta \cap (1 - \beta^2, 1) = \emptyset$.

Assume $A \in (1 - \beta, 1 - \beta^2]$. In this case (7.10) and (7.11) imply $1 < N + k < 3$, which implies $N + k = 2$. Solving (7.8) for A and using $N + k = 2$ we have

$$A = \frac{1 - k}{N} = 1 - \frac{1}{N}.$$

Since $A > 1 - \beta > 1/2$ this implies $N > 1/\beta > 2$. From (7.9) we see

$$1 + \frac{\beta^2}{1 - \beta} \geq 2A = 2 - \frac{2}{N}.$$

Rearranging this we have

$$N \leq \frac{2 - 2\beta}{1 - \beta - \beta^2}.$$

For $\beta < \frac{-1+\sqrt{13}}{6}$ we have $\frac{2-2\beta}{1-\beta-\beta^2} < 3$ and thus $N < 3$. Combined with the fact that $N > 2$, we see $\mathcal{A} \cap (1 - \beta, 1 - \beta^2] = \emptyset$ for $\beta < \frac{-1+\sqrt{13}}{6}$. Next, assume $\frac{-1+\sqrt{13}}{6} \leq \beta < 1/2$. Then

$\frac{2-2\beta}{1-\beta-\beta^2} < 4$ and we must have $N = 3$, $A = \frac{2}{3}$ and $k = -1$. It is clear that (1.2) holds. For (1.3), we use the fact that $\beta \geq \frac{-1+\sqrt{13}}{6}$ to see

$$C = 1 + \frac{\beta^2}{1-\beta} \geq \frac{4}{3} = (N+k)A.$$

Thus, by Theorem 1.1, for $\beta \geq \frac{-1+\sqrt{13}}{6}$ we have $2/3 \in \mathcal{A}_\beta$. Since $\{d_i\}$ is symmetric about $1/2$, we see that $\mathcal{A}_\beta \cap (0, \beta] = \{1/3\}$ for $\frac{-1+\sqrt{13}}{6} \leq \beta < 1/2$ and the set is empty for $\beta < \frac{-1+\sqrt{13}}{6}$. \square

In the above example, note that for any choice of β , we have $C - D \in \mathbb{Z}$ for any choice of $A \in (0, 1)$. Thus, Theorem 2.3 implies that there is a projection with diagonal $\{d_i\}$. However, if $\beta < 1/3$ then there is no $A \in (0, 1)$ so that $\{d_i\}$ is the diagonal of a self-adjoint operator E with $\sigma(E) = \{0, A, 1\}$. The next example is not the diagonal of any projection, but we will show that it is the diagonal of many different operators with three point spectrum.

Example 4. Let $\{d_i\}_{i \in \mathbb{Z}}$ be given by

$$d_i = \begin{cases} 2^{i-1} & i \leq 0 \\ 1 - 2^{-i-1} & i > 0. \end{cases}$$

Let

$$\mathcal{A} = \{A \in (0, 1) : \exists E \geq 0 \text{ with } \sigma(E) = \{0, A, 1\} \text{ and diagonal } \{d_i\}\}.$$

We claim that

$$\mathcal{A} = \left\{ \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8} \right\}.$$

The sequence $\{d_i\}$ is symmetric about $1/2$, and thus $A \in \mathcal{A}$ implies $1 - A \in \mathcal{A}$. Hence, it is enough to show that

$$\mathcal{A} \cap [\frac{1}{2}, 1) = \left\{ \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8} \right\}.$$

Assume $A \in \mathcal{A} \cap (1 - 2^{-m}, 1 - 2^{-m-1}]$ for some $m \geq 1$. We have

$$C = m - \frac{1}{2} + \frac{1}{2^m} \quad \text{and} \quad D = \frac{1}{2^m}.$$

Since $A \in \mathcal{A}$ Theorem 5.2 implies that there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$\boxed{\text{ex2.1}} \quad (7.12) \quad C - D = m - \frac{1}{2} = NA + k$$

$$\boxed{\text{ex2.2}} \quad (7.13) \quad C = m - \frac{1}{2} + 2^{-m} \geq (N+k)A.$$

Using (7.12) and $A \leq 1 - 2^{-m-1}$ we have

$$\boxed{\text{ex2.3}} \quad (7.14) \quad m-1 < m - \frac{1}{2} + N2^{-m-1} = NA + k + N2^{-m-1} \leq N(1 - 2^{-m-1}) + k + N2^{-m-1} = N + k.$$

From (7.13) and $A > 1 - 2^{-m}$ we have

$$m - \frac{1}{2} + 2^{-m} \geq (N+k)A > (N+k)(1 - 2^{-m}).$$

Rearranging gives

$$\boxed{\text{ex2.4}} \quad (7.15) \quad N + k < \left(m - \frac{1}{2} + 2^{-m}\right) \frac{2^m}{2^m - 1} = m + \frac{m - 2^{m-1} + 1}{2^m - 1}.$$

For $m \geq 3$, a simple calculation shows $\frac{m-2^{m-1}+1}{2^m-1} \leq 0$ and thus $N + k < m$. However, from (7.14) we have $N + k > m - 1$. Since $N + k \in \mathbb{Z}$ this is a contradiction and shows that $\mathcal{A} \cap (1 - 2^{-m}, 1 - 2^{-m-1}] = \emptyset$ for $m \geq 3$.

One can easily check that $A = 1/2$ satisfies (1.2) and (1.3) with $N = 1$ and $k = -1$ (or $N = 3$ and $k = -2$). All that is left is to find $\mathcal{A} \cap (1 - 2^{-m}, 1 - 2^{-m-1}]$ for $m = 1$ and 2 . The calculation for each m is similar, so the case of $m = 2$ will be left to the reader.

Assume $A \in \mathcal{A} \cap (1/2, 3/4]$. In this case we have $C = 1$ and $D = 1/2$. From (7.14) and (7.15) we have $0 < N + k < 2$ and thus $N + k = 1$. Using this and solving (7.12) for A we have

$$A = \frac{\frac{1}{2} - k}{N} = \frac{N - \frac{1}{2}}{N} = 1 - \frac{1}{2N}.$$

From the inequalities $1/2 < A = 1 - 1/(2N) \leq 3/4$ we obtain $1 < N \leq 2$. Thus $N = 2$, $A = 3/4$ and $k = -1$. One can easily check that (1.2) and (1.3) are satisfied for these values of A, N and k . \square

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